

# Partial Inertial Manifolds for infinite-dimensional dynamical systems: Example for P.D.E.s with a state-dependent delay

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**Abstract.** We propose a new notion of Partial Inertial Manifold to study the long-time asymptotic behavior of dissipative differential equations. As shown on an example, such manifolds may exist in the cases when the classical Inertial manifold does not exist (or not known to exist).

*Key words:* Partial functional differential equation, state-dependent delay, inertial manifold, partial inertial manifold.

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## 1. Introduction

Study of the long-time asymptotic behavior of solutions occupies an important place in the qualitative theory of differential equations. Considering partial and/or functional differential equations one naturally obtains infinite-dimensional dynamical systems. To investigate their asymptotic behavior many powerful methods and approaches have been developed, such as global, weak and exponential attractors [2, 7, 33, 4], inertial manifolds [8, 9, 33, 10, 11], approximate inertial manifolds [12, 8, 33], determining functional [4] etc.

During these investigations many deep results were obtained so far and the subject continuously attracts attention of many researchers. Each of the mentioned objects (attractors, manifolds, functionals) indicates important features of the dynamical systems under considerations, but naturally has special conditions to exist. If we are able to establish simultaneously the existence of several of the mentioned objects for a system, then we get more important information on its asymptotic properties. In this note we introduce a new notion - Partial Inertial Manifold and hope it will be useful for the study.

## 2. Partial Inertial Manifolds

Consider a dynamical system  $(S(t), \mathcal{H})$ , where  $S(t) : \mathcal{H} \rightarrow \mathcal{H}$  denotes the evolution operator and  $\mathcal{H}$  is the phase space (see e.g. [2, 15, 33, 13, 4] for more details). For

example, one may consider a general dissipative differential equation in the space  $\mathcal{H}$

$$\dot{u} + Au = B(u), \quad u \in \mathcal{H}, \quad (1)$$

where  $A$  is the (leading in some sense) linear part, and  $B$  is the nonlinearity. Under the natural assumptions this equation generates an evolution operator as a shift along the trajectories of (1) i.e.  $S(t)u^0 \equiv u(t; u^0)$ , where  $u(t; u^0)$  denotes the solution of (1) with the initial data  $u(0) = u^0$ .

Such objects as global attractors and inertial manifolds play an important role in the study of long-time asymptotic behavior of dissipative dynamical systems. We recall [8, 9, 33, 10, 11]

**Definition 1.** *A set  $\mathcal{M} \subset \mathcal{H}$  is called an Inertial manifold if there exist a projector  $P = P^2 : \mathcal{H} \rightarrow \mathcal{H}$  and a Lipschitz mapping  $\Phi : P\mathcal{H} \rightarrow (1 - P)\mathcal{H}$  such that*

- $\dim P < \infty$ ;
- $\mathcal{M} = \{u : u = p + \Phi(p), \quad p \in P\mathcal{H}\} \subset \mathcal{H}$ ;
- $S(t)\mathcal{M} \subset \mathcal{M}$  for all  $t \geq 0$ ;
- for any  $u \in \mathcal{H}$  one has  $\text{dist}_{\mathcal{H}}\{S(t)u, \mathcal{M}\} \leq K(\|u\|_{\mathcal{H}}) \cdot \exp\{-\alpha t\}$  for some  $\alpha > 0$ .

The existing theory says that a dynamical system usually has an Inertial manifold provided special *spectral gap conditions* are satisfied (see e.g. [2, 33, 4] for more details). These conditions are usually formulated as a condition for the distance between two nearest eigenvalues  $|\lambda_{N+1} - \lambda_N|$  of the leading linear part of the differential equation to be big enough in comparison with the Lipschitz constant of the nonlinear part of the differential equation and (possibly) lower degrees  $\lambda_{N+1}^\alpha, \lambda_N^\alpha, \alpha \in [0, 1)$  of the eigenvalues. In this direction, to get an inertial manifold, one first computes the Lipschitz constant  $L$  of the nonlinear part  $B$  and then looks for an integer  $N$  such that  $|\lambda_{N+1} - \lambda_N| \geq C(L, \lambda_{N+1}^\alpha, \lambda_N^\alpha)$  (to be more precise, one needs to consider a concrete equation). Unfortunately, the spectral gap conditions are very restrictive and do not hold for many important problems. To investigate the cases when inertial manifold does not exist (or not known to exist) another approaches have been proposed such as approximate inertial manifolds, exponential attractors etc (see e.g. [33, 4]).

In this note we propose a new approach. The main idea is to look for a subset  $D$  of the phase space  $\mathcal{H}$  such that the restriction of the nonlinear term of the differential

equation on the set  $\bigcup_{t \geq 0} S(t)D \subset \mathcal{H}$  has a small enough Lipschitz constant. If we are able to extend the restriction of the nonlinear term from  $\bigcup_{t \geq 0} S(t)D$  to  $\mathcal{H}$  without increasing the Lipschitz constant, then we get an auxiliary nonlinear term  $B_\ell$ . If the spectral gap conditions are satisfied with this (smaller) Lipschitz constant, then equation (1) with the nonlinearity  $B_\ell$  does have an inertial manifold. This manifold is finite-dimensional and attracts all the trajectories of the initial equation (1) which start in  $\bigcup_{t \geq 0} S(t)D$ . We call this manifold *partial inertial manifold* for (1). The name reflects the fact that the manifold attracts only *part* of the phase space, but not the whole  $\mathcal{H}$ . Considerations become simpler if the set  $D$  is positively invariant i.e.  $S(t)D \subset D$  for all  $t \geq 0$ , then  $\bigcup_{t \geq 0} S(t)D = D$ .

We summarize the above ideas in the following

**Definition 2.** *A set  $\mathcal{M} \subset \mathcal{H}$  is called a Partial Inertial Manifold if there exist a projector  $P = P^2 : \mathcal{H} \rightarrow \mathcal{H}$ , a Lipschitz mapping  $\Phi : P\mathcal{H} \rightarrow (1 - P)\mathcal{H}$  and a set  $D \subset \mathcal{H}$  such that*

- $\dim P < \infty$ ;
- $\mathcal{M} = \{u : u = p + \Phi(p), \quad p \in P\mathcal{H}\} \subset \mathcal{H}$ ;
- *for any  $u \in D \subset \mathcal{H}$  one has  $\text{dist}_{\mathcal{H}}\{S(t)u, \mathcal{M}\} \leq K(\|u\|_{\mathcal{H}}) \cdot \exp\{-\alpha t\}$  for some  $\alpha > 0$ .*

**Remark.** *It is easy to see that Definition 2 gives the possibility to exist more than one Partial Inertial Manifolds for the same equation if we have several sets  $D_i$  with the described properties. On the other hand, the classical Inertial Manifold is a Partial Inertial Manifold if we set  $D = \mathcal{H}$ .*

In the next section we present a concrete example of a system of partial differential equations with state-dependent distributed delay for which a partial inertial manifold exists while inertial manifold does not. The construction of the example is based on our recent studying of P.D.E.s with state-dependent delay [27, 28]. For more details on state-dependent (ordinary) equations see e.g. [22, 36].

### 3. Example of the existence of a P.I.M.: state-dependent delay equations

Consider the following partial differential equation with state-dependent distributed delay

$$\frac{\partial}{\partial t}u(t, x) + Au(t, x) = \int_{-r}^0 b(u(t + \theta, x))\xi(\theta, u_t)d\theta \equiv (B_1[\xi](u_t))(x), \quad x \in \Omega, \quad (2)$$

where  $A$  is a densely-defined self-adjoint positive linear operator with domain  $D(A) \subset L^2(\Omega)$  and with compact resolvent, so  $A : D(A) \rightarrow L^2(\Omega)$  generates an analytic semi-group,  $\Omega$  is a smooth bounded domain in  $R^{n_0}$ ,  $b : R \rightarrow R$  is a locally Lipschitz bounded map ( $|b(w)| \leq M_b$  with  $M_b \geq 0$ ), The function  $\xi(\cdot, \cdot) : [-r, 0] \times C \rightarrow R$  represents the state-dependent distributed delay. We denote for short  $C \equiv C([-r, 0]; L^2(\Omega))$ . As usually for delay equations, we denote by  $u_t$  the function of  $\theta \in [-r, 0]$  by the formula  $u_t \equiv u_t(\theta) \equiv u(t + \theta)$ . For more details on delay equations we refer to the classical monographs [15, 13, 37, 17].

We consider equation (2) with the following initial conditions

$$u|_{[-r, 0]} = \varphi \in C \equiv C([-r, 0]; L^2(\Omega)). \quad (3)$$

The methods used in our work can be applied to another types of nonlinear and delay PDEs. We choose a particular form of nonlinear delay term  $B_1$  for simplicity and to illustrate our approach on the diffusive Nicholson's blowflies equation (see below for more details).

Assume the following:

$$\mathbf{A1)} \quad |b(s)| \leq M_b \text{ and } |b(s^1) - b(s^2)| \leq L_b |s^1 - s^2|, \text{ for all } s, s^1, s^2 \in R. \quad (4)$$

$$\mathbf{A2)} \quad \int_{-r}^0 |\xi(\theta, \psi^1) - \xi(\theta, \psi^2)| d\theta \leq L_{\xi, M}^{1,1} \cdot \|\psi^1 - \psi^2\|_{L^1(-r, 0; L^1(\Omega))}, \quad (5)$$

$$\mathbf{A3)} \quad \text{ess sup}_{\theta \in (-r, 0)} |\xi(\cdot, \psi)| \leq M_\xi \text{ for all } \psi \in C. \quad (6)$$

We notice that assumptions (4)-(6) are more restrictive than the ones of [28, theorems 1,2], so we can apply theorems 1,2 from [28] to get the existence and uniqueness of solutions for (2), (3) with  $\varphi \in C$ . In this note we are interested in continuous solutions i.e. functions  $u \in C([0, T]; L^2(\Omega))$  for any  $T > 0$ .

In the same manner, using [28, theorems 1,2], we define an *evolution operator*  $S_t : C \rightarrow C$  by the formula  $S_t = u_t(\varphi)$ , where  $u(\varphi)$  denotes the unique (continuous) solution of (2),(3) with the initial condition  $u_0(\varphi) = \varphi$ . Sometimes, we will write  $S_t[\xi]$  to indicate the kernel function  $\xi$  in the nonlinearity  $B_1[\xi]$  (see (2)).

Notice that due to the inclusion  $C \subset L^1(-r, 0; L^1(\Omega))$ , we get for any  $v^1, v^2 \in C$ :

$$\begin{aligned} \|v^1 - v^2\|_{L^1(-r, 0; L^1(\Omega))} &= \int_{-r}^0 \left\{ \int_{\Omega} |v^1(\theta, x) - v^2(\theta, x)| dx \right\} d\theta \\ &\leq \sqrt{|\Omega|} \int_{-r}^0 \|v^1(\theta, \cdot) - v^2(\theta, \cdot)\|_{L^2(\Omega)} d\theta \leq r \cdot \sqrt{|\Omega|} \cdot \|v^1 - v^2\|_C. \end{aligned} \quad (7)$$

Hence (5) implies

$$\int_{-r}^0 |\xi(\theta, \psi^1) - \xi(\theta, \psi^2)| d\theta \leq L_{\xi, M}^{1,1} \cdot r \cdot \sqrt{|\Omega|} \cdot \|v^1 - v^2\|_C. \quad (8)$$

Let us check that the mapping  $B_1 \equiv B_1[\xi] : C \rightarrow L^2(\Omega)$  satisfies the Lipschitz property (c.f. (2.3) in [3]). Using (8), one has

$$\begin{aligned} \|B_1(v_0^1) - B_1(v_0^2)\|^2 &= \int_{\Omega} \left| \int_{-r}^0 \{b(v^1(\theta, x))\xi(\theta, v_0^1) - b(v^2(\theta, x))\xi(\theta, v_0^2)\} d\theta \right|^2 dx \\ &\leq 2L_b^2 M_{\xi}^2 \int_{\Omega} \left( \int_{-r}^0 |v^1(\theta, x) - v^2(\theta, x)| d\theta \right)^2 dx + 2M_b^2 \left( L_{\xi, M}^{1,1} \right)^2 r^2 |\Omega| \cdot \|v^1 - v^2\|_C^2 \\ &\leq 2L_b^2 M_{\xi}^2 r \int_{\Omega} \left\{ \int_{-r}^0 |v^1(\theta, x) - v^2(\theta, x)|^2 d\theta \right\} dx + 2M_b^2 \left( L_{\xi, M}^{1,1} \right)^2 r^2 |\Omega| \cdot \|v^1 - v^2\|_C^2 \\ &\leq 2L_b^2 M_{\xi}^2 r \int_{-r}^0 \|v^1(\theta, \cdot) - v^2(\theta, \cdot)\|_{L^2(\Omega)}^2 d\theta + 2M_b^2 \left( L_{\xi, M}^{1,1} \right)^2 r^2 |\Omega| \cdot \|v^1 - v^2\|_C^2 \\ &\leq 2 \left( L_b^2 M_{\xi}^2 r^2 + M_b^2 \left( L_{\xi, M}^{1,1} \right)^2 r^2 |\Omega| \right) \cdot \|v^1 - v^2\|_C^2. \end{aligned}$$

So, we get (c.f. (2.3) in [3])

$$\|B_1(v^1) - B_1(v^2)\| \leq M_1 \cdot \|v^1 - v^2\|_C \quad \text{with} \quad M_1 \equiv r \cdot \sqrt{2 \left( L_b^2 M_{\xi}^2 + M_b^2 \left( L_{\xi, M}^{1,1} \right)^2 |\Omega| \right)}. \quad (9)$$

To get estimate (9), we used the following calculations

$$\begin{aligned} &\left| \int_{-r}^0 \{b(v^1(\theta, x))\xi(\theta, v_0^1) - b(v^2(\theta, x))\xi(\theta, v_0^2)\} d\theta \right| \\ &\leq L_b \int_{-r}^0 |v^1(\theta, x) - v^2(\theta, x)| \cdot |\xi(\theta, v_0^1)| d\theta + M_b \int_{-r}^0 |\xi(\theta, v_0^1) - \xi(\theta, v_0^2)| d\theta \\ &\leq L_b M_{\xi} \int_{-r}^0 |v^1(\theta, x) - v^2(\theta, x)| d\theta + M_b L_{\xi, M}^{1,1} \cdot r \cdot \sqrt{|\Omega|} \cdot \|v_0^1 - v_0^2\|_C \end{aligned}$$

and the inclusion  $C \subset L^2((-r, 0) \times \Omega)$ , which implies  $\int_{-r}^0 |v^1(\theta, x) - v^2(\theta, x)| d\theta \leq \sqrt{r} \cdot \left( \int_{-r}^0 |v^1(\theta, x) - v^2(\theta, x)|^2 d\theta \right)^{\frac{1}{2}}$  and, as a result,  $\left( \int_{-r}^0 |v^1(\theta, x) - v^2(\theta, x)| d\theta \right)^2 \leq r \cdot \int_{-r}^0 |v^1(\theta, x) - v^2(\theta, x)|^2 d\theta$ .

Now we recall a sufficient conditions for the existence of an inertial manifold in the case of delay semilinear parabolic equations [3].

Since  $A : D(A) \subset L^2(\Omega) \rightarrow L^2(\Omega)$  is a densely-defined self-adjoint positive linear operator, then there exists an orthonormal basis  $\{e_k\}$  of  $L^2(\Omega)$  such that

$$Ae_k = \lambda_k e_k, \quad \text{with} \quad 0 < \lambda_1 \leq \lambda_2 \leq \dots, \quad \lim_{k \rightarrow \infty} \lambda_k = \infty.$$

As in [3], we fix an integer  $N$  and denote  $P = P_N$  the orthogonal projector onto the space spanned by the first  $N$  eigenvectors of  $A$ . We also define the  $N$ -dimensional projector  $\hat{P} = \hat{P}_N$  in  $C$  by

$$\hat{P}\phi = (\hat{P}\phi)(\theta) = \sum_{k=1}^N e^{-\lambda_k \theta} \langle \phi(0), e_k \rangle_{L^2(\Omega)} \cdot e_k \equiv e^{-A\theta} \phi(0), \quad \phi \in C, \quad \theta \in [-r, 0].$$

From the above considerations we see that one can apply theorem 3.1 from [3] to the system (2) under the following assumptions (see [3]):

**A4)** For some  $N$  and  $\mu > 0$  the following *spectral gap condition* is satisfied  $\lambda_{N+1} - \lambda_N \geq 2\mu$  (see (2.8) in [3]);

**A5)** Constants  $\mu, N$  and delay  $r > 0$  satisfy:  $\mu > 4M_1$  and  $\delta \equiv \frac{2}{\mu} M_1 \cdot e^{(\lambda_N + \mu)r} \leq \frac{1}{2}$  (see (3.1) in [3]).

These two assumptions give (theorem 3.1 from [3]) the existence of the  $N$ -dimensional asymptotically complete manifold (inertial manifold)

$$\mathcal{M} = \{\hat{p}(\theta) + \Phi(\hat{p}(0), \theta) : \hat{p}(\theta) \in \hat{P}C\} \subset C$$

which is invariant for solutions of (2), (3). Here  $\Phi$  is a Lipschitz map  $\Phi : PL^2(\Omega) \rightarrow (1 - \hat{P})C$ .

If we choose the biggest possible value of constant  $\mu = \frac{1}{2}(\lambda_{N+1} - \lambda_N)$ , then we get an estimate for the upper bound of the Lipschitz constant  $M_1$ :

$$M_1 \leq \frac{\lambda_{N+1} - \lambda_N}{8} \cdot \exp \left\{ -\frac{(\lambda_{N+1} + \lambda_N)}{2} \cdot r \right\}. \quad (10)$$

Our goal is to illustrate that in the case when (10) does not hold, it is possible that partial inertial manifolds do exist.

### 3.1. Construction of the kernel function $\xi$ .

Let us choose

$$\xi^+(\theta) \geq 0 \text{ a. e. in } \theta \in (-r, 0) \quad \text{and} \quad \xi^-(\theta) \leq 0 \text{ a. e. in } \theta \in (-r, 0) \quad (11)$$

such that

$$\operatorname{ess\,sup}_{\theta \in (-r, 0)} |\xi^\pm(\theta)| \leq \frac{1}{2} M_\xi. \quad (12)$$

For any  $v \in C$  we write

$$v(\theta, x) = v^+(\theta, x) + v^-(\theta, x) \quad (13)$$

where (a.e. in  $x \in \Omega$ )

$$v^+(\theta, x) \equiv \sup\{v(\theta, x), 0\} \geq 0, \quad v^-(\theta, x) \equiv \inf\{v(\theta, x), 0\} \leq 0. \quad (14)$$

We will use the following property

$$\|v\|_{L^1(-r,0;L^1(\Omega))} = \|v^+\|_{L^1(-r,0;L^1(\Omega))} + \|v^-\|_{L^1(-r,0;L^1(\Omega))}. \quad (15)$$

Now we are ready to define for any  $v \in C$

$$\xi(\theta, v) = \xi^+(\theta) \cdot \min\{\|v^+\|_{L^1(-r,0;L^1(\Omega))}, 1\} + \xi^-(\theta) \cdot \min\{\|v^-\|_{L^1(-r,0;L^1(\Omega))}, 1\}. \quad (16)$$

Using the property (for any norm  $\|\cdot\|$ )

$$\min\{\|\psi^1\|, 1\} - \min\{\|\psi^2\|, 1\} \leq \|\psi^1\| - \|\psi^2\| \leq \|\psi^1 - \psi^2\|, \quad (17)$$

one can check that  $\xi$ , defined in (16), satisfies (5) with

$$L_{\xi,M}^{1,1} \equiv \max\left\{\int_{-r}^0 |\xi^+(\theta)| d\theta, \int_{-r}^0 |\xi^-(\theta)| d\theta\right\}. \quad (18)$$

More precisely (we will write  $\|\cdot\|_{L^{1,1}} \equiv \|\cdot\|_{L^1(-r,0;L^1(\Omega))}$  for short):

$$\begin{aligned} \int_{-r}^0 |\xi(\theta, v^1) - \xi(\theta, v^2)| d\theta &= \int_{-r}^0 |\xi^+(\theta) \cdot [\min\{\|v^{1+}\|_{L^{1,1}}, 1\} - \min\{\|v^{2+}\|_{L^{1,1}}, 1\}] \\ &\quad + \xi^-(\theta) \cdot [\min\{\|v^{1-}\|_{L^{1,1}}, 1\} - \min\{\|v^{2-}\|_{L^{1,1}}, 1\}]| d\theta \\ &\leq \int_{-r}^0 \left\{ |\xi^+(\theta)| \cdot \|v^{1+} - v^{2+}\|_{L^{1,1}} + |\xi^-(\theta)| \cdot \|v^{1-} - v^{2-}\|_{L^{1,1}} \right\} d\theta \\ &\leq \max\left\{\int_{-r}^0 |\xi^+(\theta)| d\theta, \int_{-r}^0 |\xi^-(\theta)| d\theta\right\} \cdot (\|v^{1+} - v^{2+}\|_{L^{1,1}} + \|v^{1-} - v^{2-}\|_{L^{1,1}}) \\ &\leq L_{\xi,M}^{1,1} \cdot \|v^1 - v^2\|_{L^{1,1}}, \end{aligned}$$

where  $L_{\xi,M}^{1,1}$  is defined by (18). Here we also use (15).

Definition (16) and assumption (12) give (6). Hence we conclude that function  $\xi$ , defined by (16), satisfies assumptions (5), (6).

### 3.2. Properties of the delay term $B_1[\xi]$ .

Let us define  $D_+ \equiv \{v \in C : \forall \theta \in [-r, 0] \Rightarrow v(\theta, x) \geq 0 \text{ a. e. in } x \in \Omega\} \subset C$  and  $D_- \equiv \{v \in C : \forall \theta \in [-r, 0] \Rightarrow v(\theta, x) \leq 0 \text{ a. e. in } x \in \Omega\} \subset C$ .

In addition to (4), we assume that function  $b$  satisfies

$$\mathbf{A6)} \quad b(s) = b(-s) \geq 0, \quad s \in R. \quad (19)$$

So definitions (16), (11) and assumption (19) give

$$\forall v \in D_+ \Rightarrow B_1(v) \geq 0 \quad \text{a. e. in } x \in \Omega, \quad \text{and} \quad \forall v \in D_- \Rightarrow B_1(v) \leq 0 \quad \text{a. e. in } x \in \Omega.$$

The last property implies (see [16]) that cones  $D_+, D_-$  are positively invariant i.e.

$$S_t[\xi]D_+ \subset D_+ \quad \text{and} \quad S_t[\xi]D_- \subset D_-. \quad (20)$$

Here  $S_t[\xi] : C \rightarrow C$  denotes the evolution operator constructed by the solutions of (2), (3) with the kernel function  $\xi$  in (2), defined by (16).

Now we consider two auxiliary functions (see (16), (13), (14))

$$\xi^p(\theta, v) \equiv \xi^+(\theta) \cdot \min \left\{ \|v^+\|_{L^1(-r,0;L^1(\Omega))}, 1 \right\}, \quad (21)$$

$$\xi^n(\theta, v) \equiv \xi^-(\theta) \cdot \min \left\{ \|v^-\|_{L^1(-r,0;L^1(\Omega))}, 1 \right\}. \quad (22)$$

Since  $\forall v \in D_+ \Rightarrow \xi^p(\theta, v) = \xi(\theta, v)$ , then property (20) gives

$$\forall v \in D_+ \Rightarrow S_t[\xi^p]v = S_t[\xi]v. \quad (23)$$

In the same way,  $\forall v \in D_- \Rightarrow S_t[\xi^n]v = S_t[\xi]v$ .

The above considerations clearly show that  $B_1[\xi^p]$  satisfies (5) with the Lipschitz constant  $M_1 = M_1[\xi^p]$  defined by (9) where the constant  $L_{\xi^p,M}^{1,1} = \int_{-r}^0 |\xi^+(\theta)|d\theta$  instead of  $L_{\xi,M}^{1,1} = \max \left\{ \int_{-r}^0 |\xi^+(\theta)|d\theta, \int_{-r}^0 |\xi^-(\theta)|d\theta \right\}$  (see (18)). In the same manner, we get the Lipschitz constant for  $B_1[\xi^n]$  by (9) with  $L_{\xi^n,M}^{1,1} = \int_{-r}^0 |\xi^-(\theta)|d\theta$ .

Due to the explicit dependence of the Lipschitz constants  $M_1 = M_1[\xi^p]$  and  $M_1 = M_1[\xi^n]$  on the values  $\int_{-r}^0 |\xi^+(\theta)|d\theta, \int_{-r}^0 |\xi^-(\theta)|d\theta$  (see (9)), we may choose small enough value of  $\int_{-r}^0 |\xi^+(\theta)|d\theta$  and big enough value of  $\int_{-r}^0 |\xi^-(\theta)|d\theta$  such that the constant  $M_1[\xi^p]$  satisfies (10) while  $M_1[\xi^n]$  does not. Of course, we also need the value  $rL_bM_\xi$  to be small enough (see (9)). In this case, by (18), the constant  $M_1[\xi]$  does not satisfy (10).

**Remark.** *More precisely, Let us first choose and fix  $r$  small enough to satisfy (see (10))*

$$r \leq \frac{\lambda_{N+1} - \lambda_N}{16L_bM_\xi} \cdot \exp \left\{ -\frac{\lambda_{N+1} + \lambda_N}{2} \cdot r \right\}. \quad (24)$$

*Then, for the fixed value of  $r$ , choose  $\xi^+(\cdot)$  such that*

$$\int_{-r}^0 |\xi^+(\theta)|d\theta \leq \frac{\lambda_{N+1} - \lambda_N}{16rM_b\sqrt{|\Omega|}} \cdot \exp \left\{ -\frac{\lambda_{N+1} + \lambda_N}{2} \cdot r \right\}. \quad (25)$$



Assumptions (24), (25) imply that  $M_1[\xi^p]$  satisfies (10). Now we choose  $\xi^-(\cdot)$  such that

$$\int_{-r}^0 |\xi^-(\theta)| d\theta > \frac{\lambda_{N+1} - \lambda_N}{8 r M_b \sqrt{|\Omega|}} \cdot \exp \left\{ -\frac{\lambda_{N+1} + \lambda_N}{2} \cdot r \right\}. \quad (26)$$

Assumptions (24), (26) imply that  $M_1[\xi]$  and  $M_1[\xi^n]$  do not satisfy (10).

These considerations clearly show that the system (2), (3) with the right hand side  $B_1[\xi]$  ( $\xi$  defined by (16)) does not possess an inertial manifold, while the system (2), (3) with the right hand side  $B_1[\xi^p]$  ( $\xi^p$  defined by (21)) does possess (due to [3, theorem 3.1]). Since the evolution operators  $S_t[\xi^p]$  and  $S_t[\xi]$  coincide on  $D_+$  (see (23)), we may conclude that the system (2), (3) with the right hand side  $B_1[\xi]$  ( $\xi$  defined by (16)) possesses a finite-dimensional manifold (inertial manifold for the system with  $B_1[\xi^p]$ ) which exponentially attracts all the trajectories starting in  $v \in D_+$ . This is a *partial inertial manifold* for the system (2), (3) with  $B_1[\xi]$ .

As an application we can consider the diffusive Nicholson's blowflies equation (see e.g. [30, 32]) with state-dependent delay [27, 28]. More precisely, we consider equation (2) where  $-A$  is the Laplace operator with the Dirichlet boundary conditions,  $\Omega \subset \mathbb{R}^{n_0}$  is a bounded domain with a smooth boundary, the nonlinear function  $b$  is given by  $b(w) = p \cdot w^2 e^{-|w|}$ . As a result, we conclude that under the above assumptions, the diffusive Nicholson's equation possesses a partial inertial manifold.

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## References

- [1] N.V. Azbelev, V.P. Maksimov and L.F. Rakhmatullina, Introduction to the theory of functional differential equations, Moscow, Nauka, 1991.
- [2] A. V. Babin, and M. I. Vishik, Attractors of Evolutionary Equations, Amsterdam, North-Holland, 1992.
- [3] L. Boutet de Monvel, I. D. Chueshov and A. V. Rezounenko, Inertial manifolds for retarded semilinear parabolic equations, *Nonlinear Analysis*, 34 (1998), 907-925.

- [4] I. D. Chueshov, Introduction to the Theory of Infinite-Dimensional Dissipative Systems, Acta, Kharkov (1999), (in Russian). English transl. Acta, Kharkov (2002) (see <http://www.emis.de/monographs/Chueshov> ).
- [5] I. D. Chueshov, On a certain system of equations with delay, occuring in aeroelasticity, J. Soviet Math. **58**, 1992, p.385-390.
- [6] I. D. Chueshov, A. V. Rezounenko, Global attractors for a class of retarded quasilinear partial differential equations, C.R.Acad.Sci.Paris, Ser.I **321** (1995), 607-612, ( detailed version: Math.Physics, Analysis, Geometry, Vol.2, N.3 (1995), 363-383).
- [7] Eden A., Foias C., Nicolaenko B., Temam R., Exponential Attractors for Dissipative Evolution Equations, Masson, Paris, Collection Recherches au Mathematiques Appliquees, 1994.
- [8] Foias C., Sell G., Temam R., Variétés Inertiellles des équations différentielles dissipatives, C. R. Acad. Sci. Paris, Serie I., 301 (1985) 139-142.
- [9] Foias C., Sell G., Titi E., Exponential tracking and approximation of inertial manifolds for dissipative equations, J. Dyn. Diff. Eqns., 1 (1989) 199-224.
- [10] Chow S.-N., Lu K., Invariant manifolds for flows in Banach spaces, J. Diff. Eqns., 74 (1988) 285-317.
- [11] Constantin P., Foias C., Nicolaenko B., Temam R., Integral Manifolds and Inertial Manifolds for Dissipative Partial Differential Equations, Springer, Berlin, 1989.
- [12] Foias C., Manley O., Temam R., Sur l'interaction des petits et grands tourbillons dans les écoulements turbulents, C.R. Acad. Sci. Paris, Serie I., 305 (1987) 497 - 500.
- [13] O. Diekmann, S. van Gils, S. Verduyn Lunel, H-O. Walther, Delay Equations: Functional, Complex, and Nonlinear Analysis, Springer-Verlag, New York, 1995.
- [14] J. K. Hale, Theory of Functional Differential Equations, Springer, Berlin-Heidelberg- New York, 1977.

- [15] J. K. Hale and S. M. Verduyn Lunel, Theory of Functional Differential Equations, Springer-Verlag, New York, 1993.
- [16] D. Henry, Geometric theory of semilinear parabolic equations. Lecture Notes in Mathematics, 840. Springer-Verlag, Berlin-New York, 1981.
- [17] T. Krisztin, H.-O. Walther and J. Wu, Shape, Smoothness and Invariant Stratification of an Attracting Set for Delayed Monotone Positive Feedback, Fields Institute Monographs, 11, AMS, Providence, RI, 1999.
- [18] J. L. Lions, Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires, Dunod, Paris, 1969.
- [19] A.D. Mishkis, Linear differential equations with retarded argument. 2nd edition, Nauka, Moscow, 1972.
- [20] J. Mallet-Paret and R. D. Nussbaum, Boundary layer phenomena for differential-delay equations with state-dependent time lags I, Archive for Rational Mechanics and Analysis 120 (1992), 99-146.
- [21] J. Mallet-Paret and R. D. Nussbaum, Boundary layer phenomena for differential-delay equations with state-dependent time lags II, J. Reine Angew. Math., 477 (1996), 129-197.
- [22] J. Mallet-Paret, R. D. Nussbaum, P. Paraskevopoulos, Periodic solutions for functional-differential equations with multiple state-dependent time lags, Topol. Methods Nonlinear Anal. 3 (1994), no. 1, 101–162.
- [23] A. V. Rezounenko, On singular limit dynamics for a class of retarded nonlinear partial differential equations, Matematicheskaya fizika, analiz, geometriya, 4 (1/2), (1997), 193-211.
- [24] A.V. Rezounenko, Inertial manifolds with delay for retarded semilinear parabolic equations, Discr. Contin. Dynamical Systems, 6 (2000), 829-840.
- [25] A.V. Rezounenko, Approximate inertial manifolds for retarded semilinear parabolic equations, Journal of Mathematical Analysis and Applications, 282 (2) (2003), 614-628.
- [26] A.V. Rezounenko, A short introduction to the theory of ordinary delay differential equations. Lecture Notes. Kharkov University Press, Kharkov, 2004.

- [27] A.V. Rezounenko, J. Wu, A non-local PDE model for population dynamics with state-selective delay: local theory and global attractors, *Journal of Computational and Applied Mathematics*, 190, Issues 1-2 (2006), P.99-113.
- [28] Rezounenko A.V., Partial differential equations with discrete and distributed state-dependent delays, *Journal of Mathematical Analysis and Applications*, 326, Issue **2**, (2007), 1031-1045. ( see preprint version: "Rezounenko A.V., Two models of partial differential equations with discrete and distributed state-dependent delays". *preprint*. March 22, 2005, <http://arxiv.org/abs/math.DS/0503470> ).
- [29] R.E. Showalter, Monotone operators in Banach space and nonlinear partial differential equations, AMS, Mathematical Surveys and Monographs, vol. 49, 1997.
- [30] J. W. -H. So, J. Wu and Y. Yang, Numerical steady state and Hopf bifurcation analysis on the diffusive Nicholson's blowflies equation. *Appl. Math. Comput.* 111 (2000), no. 1, 33–51.
- [31] J. W. -H. So, J. Wu and X.Zou, A reaction diffusion model for a single species with age structure. I. Travelling wavefronts on unbounded domains, *Proc. Royal Soc. Lond. A* (2001) 457, 1841-1853.
- [32] J. W.- H. So and Y. Yang, Dirichlet problem for the diffusive Nicholson's blowflies equation, *J. Differential Equations* 150 (1998), no. 2, 317–348.
- [33] R. Temam, Infinite Dimensional Dynamical Systems in Mechanics and Physics, Springer, Berlin-Heidelberg-New York, 1988.
- [34] C. C. Travis and G. F. Webb, Existence and stability for partial functional differential equations, *Transactions of AMS* **200**, (1974), 395-418.
- [35] H. -O. Walther, Stable periodic motion of a system with state dependent delay, *Differential and Integral Equations* 15 (2002), 923-944.
- [36] H.-O. Walther, The solution manifold and  $C^1$ -smoothness for differential equations with state-dependent delay, *J. Differential Equations* 195 (2003), no. 1, 46–65.

- [37] J. Wu, Theory and Applications of Partial Functional Differential Equations, Springer-Verlag, New York, 1996.
- [38] K. Yosida, Functional analysis, Springer-Verlag, New York, 1965.

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